Algorithms for graphs MATH.APP.270

In-Person Exam 27.2.2025

1. In this problem you will use Algorithm 1, which is presented below. The inputs to Algorithm 1 are a digraph (V, E) and one vertex $s \in V$. Algorithm 1 changes the color of some of the vertices from white to black. Set $\textsc{Adj}(u)$, used in Algorithm 1, contains all vertices adjacent to u.

Algorithm 1 Algorithm 1
input (V, E), $s \in V$

```
1: for v ∈ V do
2:     col[v] := white
3: end for
4: Stack S
5: S.PUSH(s)
6: col[s] := grey
7: while S ≠ ∅ do
8:     u = S.POP()
9:     for v ∈ ADJ(u) do
10:        if col[v] = white then
11:            S.PUSH(v)
12:            col[v] = grey
13:        end if
14:    end for
15:    col[u] = black
16: end while
```

(a) Consider the following statement:

> If there is a directed path from vertex s to vertex t in the input digraph, then all vertices in this path will be colored black.

Invent a suitable loop-invariant for the **while**-loop and with the help of the invariant, prove the statement.

(b) Consider the following statement:

> The input digraph has a directed cycle $\langle u_0, \ldots, u_n \rangle$ and a directed path from s to u_0, if and only if Algorithm 1 encounters a grey vertex at line 10.

Either prove that this statement is correct or give a counter example to prove that the statement is incorrect.

2. Imagine you are a general in an ancient empire. You control n battalions (large combat units, that is not important). Each battalion has a headquarters company that has three communications specialists; communications specialists can run fast to deploy messages to other battalions. You have your own headquarters, and you have k communications specialists.
The battalions are each stationed in a fort. There are $n+1$ forts that are situatated in villages dispersed in the countryside, and connected by roads. Each fort has at least 4 roads (but possibly more) that connect them to other forts. The roads are of different lengths.
Your task is to decide which fort would house your headquarters, so as to minimize the time it takes for your orders to reach each battalion headquarters. When you send a message, each of your own communication specialist then runs to deliver their messages to battalion leaders, who in turn may deploy *at most* three specialists to further deliver the messages.
Write a model. Discuss alternatives. As the general, you can decide how the battalion headquarters will react, where they send their messengers, etc. Explain your thinking. There is no "correct" answer here. I will be grading this based on how well you model it and consider possible alternatives. Think of it as a way for you to explain what you know about relevant algorithms

3. A finite digraph having no directed cycles is typically referred to as *DAG* (Directed Acyclic Graph).
Suppose you are given a directed graph (V, E) having n vertices. The task is to find an injection $f : V \to \{1, \ldots, n\}$, having the following property [1]:

   If $(u, v) \in E$, then $f(u) < f(v)$.

   This injection defines an ordering for the vertices usually referred to as *topological ordering*. It is known that a topological ordering exists if and only if the directed graph being analyzed is a DAG.

   (a) (i) Explain why a DAG has at least one vertex from which no edge leaves.
   (ii) Explain why a DAG has at least one vertex to which no edge enters.

   (b) Explain why no topological ordering can exist for a directed graph having (at least) one cycle.

   (c) Describe an algorithm which uses the properties given in (a) to produce a topological ordering for a DAG.

[1] An injection means that function f must assign unique integer from the set $\{1, \ldots, n\}$ to each vertex.