COMP.SGN.100 Introduction to Signal Processing,
Final Exam, 20.10.2021,
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- Own calculators can be used in the exam.
- You may take the examination paper with you.

1. Below are three systems S1-S3

   S1: $y(n) = \sqrt{x(n)}$,
   S2: $y(n) = x(n) + x(n-1) + 2x(n+1)$,
   S3: $y(n) = x(n) + 3x(n+2)$.

   Which of these systems are

   (a) causal,
   (b) memoryless,
   (c) linear.

   Justify your answers. (6p)

2. (a) Calculate the DFT of the vector $x(n) = (4, 6, 3, 3)^T$. (3p)

   (b) The signal $x(n)$ and and the impulse response $h(n)$ are the following:

   $$\begin{aligned} x(n) &= \delta(n) + \delta(n-1) - 2\delta(n-2), \\ h(n) &= \delta(n+2) - \delta(n). \end{aligned}$$

   Draw the signals $x(n)$ and $h(n)$. Calculate the signal $y(n) = h(n) * x(n)$ and draw it. (3p)

3. The signal $x(n)$ with the sampling rate 40 kHz should be converted to a signal with the sampling rate 25 kHz. Determine the steps of the conversion as a block diagram using resampling ($\boxed{\uparrow L}$ and $\boxed{\downarrow M}$) and low-pass filtering ($\boxed{H(z)}$). Specify the passband, stopband and transition band intervals of the required low-pass filters in normalized frequencies, when the frequencies on the interval $0 - 12$ kHz are to be preserved. (6p)

4. Design using the window design method a filter (i.e. find out its impulse response) satisfying the following requirements:

Stopband	[0 kHz, 17 kHz]
Passband	[19 kHz, 25 kHz]
Passband ripple	0.5 dB
Minimum stopband attenuation	28 dB
Sampling frequency	50 kHz

   Use the tables below. (6p)

5. (a) The transfer function of a causal LTI system is:

$$H(z) = \frac{1 - az^{-1}}{1 - \frac{1}{2}(bz)^{-1}},$$

where the nonzero constants $a, b \in \mathbb{R}$. Determine which values of the constants a and b make the system stable. (3p)

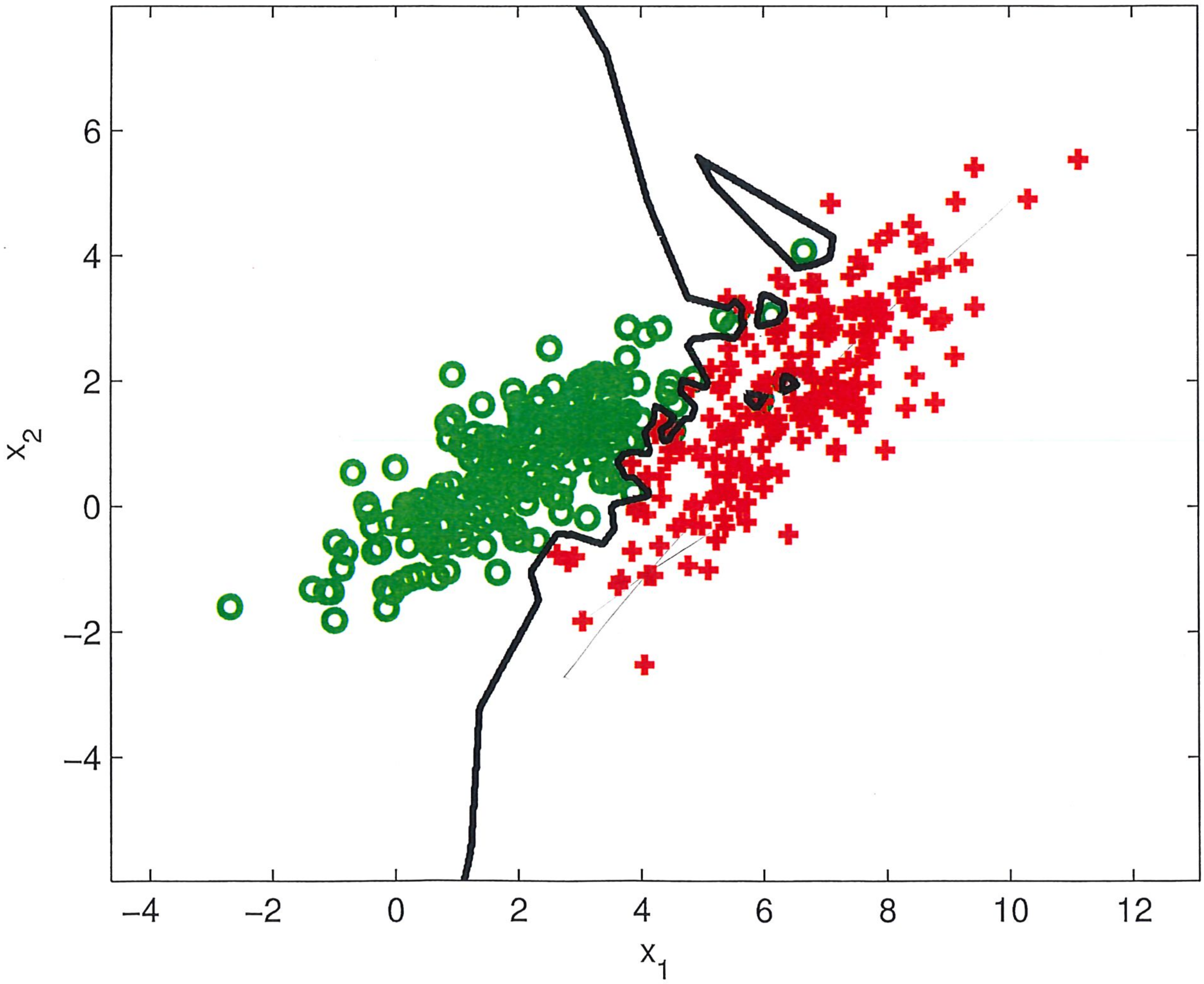
(b) Below is shown the decision boundary obtained using a classifier for a two-class classification task. Is it possible that the used classifier was

i. 1-NN classifier,

ii. 9-NN classifier,

iii. LDA classifier?

Justify your answers. (3p)



Tables

Ideal filter type	Impulse response when $n \neq 0$	$n = 0$
Low-pass	$2f_c \mathrm{sinc}(n \cdot 2\pi f_c)$	$2f_c$
High-pass	$-2f_c \mathrm{sinc}(n \cdot 2\pi f_c)$	$1 - 2f_c$
Band-pass	$2f_2 \mathrm{sinc}(n \cdot 2\pi f_2) - 2f_1 \mathrm{sinc}(n \cdot 2\pi f_1)$	$2(f_2 - f_1)$
Band-stop	$2f_1 \mathrm{sinc}(n \cdot 2\pi f_1) - 2f_2 \mathrm{sinc}(n \cdot 2\pi f_2)$	$1 - 2(f_2 - f_1)$

Name of the window function	Transition bandwidth (normalized)	Passband ripple (dB)	Minimum stopband attenuation (dB)	Window expression $w(n)$, when $\|n\| \leq (N-1)/2$
Rectangular	0.9/N	0.7416	21	1
Bartlett	3.05/N	0.4752	25	$1 - \frac{2\|n\|}{N-1}$
Hanning	3.1/N	0.0546	44	$0.5 + 0.5\cos\left(\frac{2\pi n}{N}\right)$
Hamming	3.3/N	0.0194	53	$0.54 + 0.46\cos\left(\frac{2\pi n}{N}\right)$
Blackman	5.5/N	0.0017	74	$0.42 + 0.5\cos\left(\frac{2\pi n}{N}\right) + 0.08\cos\left(\frac{4\pi n}{N}\right)$

Some Wikipedia pages that might be useful

Suppose two classes of observations have means $\vec{\mu}_0, \vec{\mu}_1$ and covariances Σ_0, Σ_1. Then the linear combination of features $\vec{w} \cdot \vec{x}$ will have means $\vec{w} \cdot \vec{\mu}_i$ and variances $\vec{w}^T \Sigma_i \vec{w}$ for $i = 0, 1$. Fisher defined the separation between these two distributions to be the ratio of the variance between the classes to the variance within the classes:

$$S = \frac{\sigma^2_{\text{between}}}{\sigma^2_{\text{within}}} = \frac{(\vec{w} \cdot \vec{\mu}_1 - \vec{w} \cdot \vec{\mu}_0)^2}{\vec{w}^T \Sigma_1 \vec{w} + \vec{w}^T \Sigma_0 \vec{w}} = \frac{(\vec{w} \cdot (\vec{\mu}_1 - \vec{\mu}_0))^2}{\vec{w}^T (\Sigma_0 + \Sigma_1) \vec{w}}$$

This measure is, in some sense, a measure of the signal-to-noise ratio for the class labelling. It can be shown that the maximum separation occurs when

$$\vec{w} \propto (\Sigma_0 + \Sigma_1)^{-1} (\vec{\mu}_1 - \vec{\mu}_0)$$

When the assumptions of LDA are satisfied, the above equation is equivalent to LDA.

Be sure to note that the vector $\vec{w}$ is the normal to the discriminant hyperplane. As an example, in a two dimensional problem, the line that best divides the two groups is perpendicular to $\vec{w}$.

Generally, the data points to be discriminated are projected onto $\vec{w}$; then the threshold that best separates the data is chosen from analysis of the one-dimensional distribution. There is no general rule for the threshold. However, if projections of points from both classes exhibit approximately the same distributions, a good choice would be the hyperplane between projections of the two means, $\vec{w} \cdot \vec{\mu}_0$ and $\vec{w} \cdot \vec{\mu}_1$. In this case the parameter c in threshold condition $\vec{w} \cdot \vec{x} > c$ can be found explicitly:

$$c = \vec{w} \cdot \frac{1}{2}(\vec{\mu}_0 + \vec{\mu}_1) = \frac{1}{2}\vec{\mu}_1^T \Sigma_1^{-1} \vec{\mu}_1 - \frac{1}{2}\vec{\mu}_0^T \Sigma_0^{-1} \vec{\mu}_0.$$

Inversion of 2 × 2 matrices [edit]

The *cofactor equation* listed above yields the following result for 2 × 2 matrices. Inversion of these matrices can be done as follows:[6]

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A more condensed form of the difference equation is:

$$y[n] = \frac{1}{a_0}\left(\sum_{i=0}^{P} b_i x[n-i] - \sum_{j=1}^{Q} a_j y[n-j]\right)$$

which, when rearranged, becomes:

$$\sum_{j=0}^{Q} a_j y[n-j] = \sum_{i=0}^{P} b_i x[n-i]$$

To find the transfer function of the filter, we first take the Z-transform of each side of the above equation, where we use the time-shift property to obtain:

$$\sum_{j=0}^{Q} a_j z^{-j} Y(z) = \sum_{i=0}^{P} b_i z^{-i} X(z)$$

We define the transfer function to be:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{P} b_i z^{-i}}{\sum_{j=0}^{Q} a_j z^{-j}}$$

Considering that in most IIR filter designs coefficient a_0 is 1, the IIR filter transfer function takes the more traditional form:

$$H(z) = \frac{\sum_{i=0}^{P} b_i z^{-i}}{1 + \sum_{j=1}^{Q} a_j z^{-j}}$$

Techniques [edit]

Conceptual approaches to sample-rate conversion include: converting to an analog continuous signal, then re-sampling at the new rate, or calculating the values of the new samples directly from the old samples. The latter approach is more satisfactory, since it introduces less noise and distortion.[3] Two possible implementation methods are as follows:

1. If the ratio of the two sample rates is (or can be approximated by)[nb 1][4] a fixed rational number *L/M*: generate an intermediate signal by inserting *L* − 1 0s between each of the original samples. Low-pass filter this signal at half of the lower of the two rates. Select every *M*-th sample from the filtered output, to obtain the result.[5]
2. Treat the samples as geometric points and create any needed new points by interpolation. Choosing an interpolation method is a trade-off between implementation complexity and conversion quality (according to application requirements). Commonly used are: ZOH (for film/video frames), cubic (for image processing) and windowed sinc function (for audio).

Equations

$$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{cases} X(n) = X_0(n) + w_N^{-n} X_1(n), & \text{when } n = 0, 1, 2, \ldots, N/2 - 1 \\ X(n) = X_0(n - N/2) + w_N^{-n} X_1(n - N/2), & \text{when } n = N/2, N/2 + 1, \ldots, N - 1 \end{cases}$$